Rotations & Spin.

Motivation & Idea.

* 1 free particle, $\psi(\vec{x})$, observables $\hat{X}$ & $\hat{P}$

Classically $\longrightarrow$ angular momentum $\vec{L} = \vec{x} \times \vec{p}$ ...

$\rightsquigarrow$ conservation of $\vec{L}$ etc ..

Quantum Mechanics $\rightarrow$ try replacing $\vec{x} \to \hat{X}$, $\vec{p} \to \hat{P}$

$\longrightarrow \hat{L} = \hat{X} \times \hat{P}$

Properties ?

$\rightarrow [L_i, L_j] = \ldots = i\hbar \varepsilon_{ijk} L_k$

$\rightarrow L^2, L_\pm$ operators ... $\rightarrow$ basis $|lm\rangle$ $l = 0, 1, 2$

* Another approach: not relying on observables $\hat{X}$ and $\hat{P}$
Say we have a rule to transform states under rotations of space $|\psi\rangle \to |\psi'\rangle = U(R)|\psi\rangle$

$U(R)$ unitary, i.e. $U(R)U(R)^\dagger = 1 = U(R)^\dagger U(R)$, $U(R)^\dagger = U(R)^{-1}$
$U(R_1 R_2) = U(R_1) U(R_2)$ (up to a phase)
$U(R(-\alpha)) = U(R(\alpha))^\dagger$
$\rightsquigarrow U(R)$ is a representation (projective) of the group of rotations $SO(3)$.

This step is where the physics comes in !!

Ex:

(a) 1 particle, $\psi(\vec{x})$, observables $\hat{X}$ & $\hat{P}$

Our rotation operation on states $|\psi\rangle \to \hat{R}|\psi\rangle$ is given by $\psi'(\vec{x}) = \psi(R^{-1}\vec{x})$

ex: $R_z(d\alpha)$:

$$\psi(R^{-1}\vec{x}) \approx \psi(x+d\alpha\, y,\ y-d\alpha\, x,\ z)$$ ≈ Taylor 1st order

$$\approx \psi(x,y,z) + \frac{\partial\psi}{\partial x}\cdot d\alpha\, y - \frac{\partial\psi}{\partial y}\, d\alpha\, x$$

$$= \underbrace{\left[1 + d\alpha\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\right]}_{\approx\, U(R_z(d\alpha))} \psi(x,y,z)$$

Note: $= iL_z$!!

(b) spin-$\frac{1}{2}$ particle, $\mathcal{H} = \mathbb{C}^2$, $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$

Observables: $\sigma_x, \sigma_y, \sigma_z$ Pauli Matrices - $[\sigma_x, \sigma_y] = i\sigma_z$ 2

└ eigenstates: $|\uparrow\rangle$ & $|\downarrow\rangle$

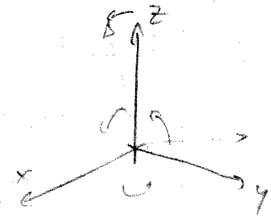
We want σ_x, σ_y & σ_z to rotate into each other under spatial rotations, !

For example: $|\uparrow\rangle \longrightarrow U(R_x(d\alpha))|\uparrow\rangle = |R\uparrow\rangle$ then we want

$$\langle R\uparrow|\underset{\vec{e}_z\cdot\vec{\sigma}}{\underset{\|}{\sigma_z}}|R\uparrow\rangle \overset{!}{=} \langle\uparrow|(R^{-1}\vec{e}_z)\cdot\vec{\sigma}|\uparrow\rangle$$

[And because $|\psi\rangle \to U(R)|\psi\rangle$, $\langle\psi|A|\psi\rangle \overset{!}{=} \langle\psi'|A'|\psi'\rangle = \langle\psi|\underbrace{U(R)^\dagger A' U(R)}|\psi\rangle$

$\longrightarrow$ $A \longrightarrow A' = U(R)\,A\,U(R)^\dagger$]



$$\langle\uparrow|(R_x^{-1}(d\alpha))\vec{e}_z|\uparrow\rangle \approx \langle\uparrow|(\sigma_z + d\alpha\,\sigma_y)|\uparrow\rangle$$

$$= \langle\uparrow|(\sigma_z + i\,d\alpha[\sigma_x, \sigma_z])|\uparrow\rangle$$

$$\approx \langle\uparrow|\underbrace{(1 + i\,d\alpha\,\sigma_x)}_{U(R)}\,\sigma_z\,\underbrace{(1 - i\,d\alpha\,\sigma_x)}_{U(R)^\dagger}|\uparrow\rangle$$

$\rightarrow$ $U(R) \approx (1 + i\,d\alpha\,\sigma_x)$

└ angular momentum operator again!

* *Message*: $\boxed{\text{Spin operators generate the rotations}}$!

Why? → Because that's how they're defined (!)

Motivated by the above examples, we *define* J_i to be the *generator* of the rotation $U(R_i)$ on the Hilbert space

$$iJ_i = \frac{d}{d\alpha} U(R_i(\alpha))\Big|_{\alpha=0}$$

So (coming from Lie group/algebra representation theory)

$$R_i(\alpha) = e^{i\alpha J_i}$$

and more generally for a rotation about $\vec{u}$

$$R_{\vec{u}}(\alpha) = e^{i\alpha \vec{u}\cdot\vec{J}} \approx (1 + i\alpha\vec{u}\cdot\vec{J})$$