# Invariance of Schrödinger Equation under Gauge Transformation.

Gauge Transformation: $\vec{A} \to \vec{A} + \vec{\nabla}\chi \qquad \phi \to \phi - \frac{1}{c}\frac{\partial \chi}{\partial t} \qquad \psi \to e^{\frac{iq}{\hbar c}\chi}\psi$

$$i\hbar\partial_t\psi \longrightarrow i\hbar\partial_t\psi' = i\hbar\left[(\partial_t\psi)e^{\frac{iq}{\hbar c}\chi} + \frac{iq}{\hbar c}(\partial_t\chi)e^{\frac{iq}{\hbar c}\chi}\psi\right] = e^{\frac{iq}{\hbar c}\chi}\left[i\hbar\partial_t\psi - \frac{q}{c}(\partial_t\chi)\psi\right]$$

$$H\psi \longrightarrow H'\psi' = \underbrace{\frac{1}{2m}(\vec{p} - \tfrac{q}{c}\vec{A}')^2\psi'}_{\circledast} + \underbrace{q\phi'\psi'}_{= e^{\frac{iq}{\hbar c}\chi}(q\phi - \frac{q}{c}(\partial_t\chi))\psi}$$

Idea: Show that

$$\circledast = e^{\frac{iq}{\hbar c}\chi}\frac{1}{2m}(\vec{p} - \tfrac{q}{c}\vec{A})^2\psi$$

Use: $[\vec{p}, e^{\frac{iq}{\hbar c}\chi}] = \frac{q}{c}(\nabla\chi)e^{\frac{iq}{\hbar c}\chi} + e^{\frac{iq}{\hbar c}\chi}\vec{p} - e^{\frac{iq}{\hbar c}\chi}\vec{p} = \frac{q}{c}(\nabla\chi)e^{\frac{iq}{\hbar c}\chi}$

Then, because $AB = [A,B] + BA$:

$$\left(\vec{p} - \tfrac{q}{c}(\vec{A} + (\vec{\nabla}\chi))\right)e^{\frac{iq}{\hbar c}\chi} = \cancel{\tfrac{q}{c}(\vec{\nabla}\chi)e^{\frac{iq}{\hbar c}\chi}} + e^{\frac{iq}{\hbar c}\chi}\vec{p} - \tfrac{q}{c}\vec{A}e^{\frac{iq}{\hbar c}\chi} - \cancel{\tfrac{q}{c}(\vec{\nabla}\chi)e^{\frac{iq}{\hbar c}\chi}}$$

$$= e^{\frac{iq}{\hbar c}\chi}(\vec{p} - \tfrac{q}{c}\vec{A}) \qquad \text{as an operator.}$$

So: $(\vec{p} - \frac{q}{c}\vec{A}')^2\psi' = (\vec{p} - \frac{q}{c}(\vec{\nabla}\chi) - \frac{q}{c}\vec{A})^2 e^{\frac{iq}{\hbar c}\chi}\psi$

$$= (\vec{p} - \tfrac{q}{c}(\vec{\nabla}\chi) - \tfrac{q}{c}\vec{A})\,e^{\frac{iq}{\hbar c}\chi}(\vec{p} - \tfrac{q}{c}\vec{A})\psi$$

$$= e^{\frac{iq}{\hbar c}\chi}(\vec{p} - \tfrac{q}{c}\vec{A})\psi \qquad \checkmark$$